

Some combinatorial sequences associated with context-free grammars*

Shi-Mei Ma [†]

School of Mathematics and Statistics, Northeastern University at Qinhuangdao,
Hebei 066004, China

Abstract

The purpose of this paper is to show that some combinatorial sequences, such as second-order Eulerian numbers and Eulerian numbers of type B , can be generated by context-free grammars.

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1 Introduction

The grammatical method was introduced by Chen [2] in the study of exponential structures in combinatorics. Let A be an alphabet whose letters are regarded as independent commutative indeterminates. A context-free grammar G over A is defined as a set of substitution rules replacing a letter in A by a formal function over A . Following Chen [2], the formal derivative D is a linear operator defined with respect to a context-free grammar G . For any formal functions u and v , we have

$$D(u + v) = D(u) + D(v), \quad D(uv) = D(u)v + uD(v) \quad \text{and} \quad D(f(u)) = \frac{\partial f(u)}{\partial u} D(u),$$

where $f(x)$ is a analytic function. By definition, we have $D^{n+1}(u) = D(D^n(u))$ for all u . For example, if $G = \{x \rightarrow xy, y \rightarrow y\}$, then

$$D(x) = xy, D(y) = y, D^2(x) = x(y + y^2), D^3(x) = x(y + 3y^2 + y^3).$$

In [5], Dumont considered chains of general substitution rules on words. It is a hot topic to explore the connection between combinatorics and context-free grammars. The reader is referred to [3, 4, 6, 11] for recent progress on this subject.

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[†]*Email address:* shimeima@yahoo.com.cn (S.-M. Ma)

We now recall some definitions, and fix some notation, that will be used throughout the rest of this paper. Let $[n] = \{1, 2, \dots, n\}$. Let \mathcal{S}_n denote the symmetric group of all permutations of $[n]$. The *Eulerian number* $\langle n \rangle_k$ enumerates the number of permutations in \mathcal{S}_n with k descents (i.e., $i < n, \pi(i) > \pi(i+1)$) as well as the number permutations in \mathcal{S}_n which have k excedances (i.e., $i < n, \pi(i) > i$) (see [12, A008292]). The numbers $\langle n \rangle_k$ satisfy the recurrence relation

$$\langle n \rangle_k = (k+1) \langle n-1 \rangle_k + (n-k) \langle n-1 \rangle_{k-1},$$

the initial condition $\langle 0 \rangle = 1$ and boundary conditions $\langle 0 \rangle_k = 0$ for $k \geq 1$. Let

$$A_n(t) = \sum_{k=0}^{n-1} \langle n \rangle_k t^k$$

be the *Eulerian polynomial*. The exponential generating function for $A_n(t)$ is

$$A(t, z) = 1 + \sum_{n \geq 1} t A_n(t) \frac{z^n}{n!} = \frac{1-t}{1-te^{z(1-t)}}. \quad (1)$$

We now consider a restricted version of Eulerian numbers. Let r be a nonnegative integer. Denote by $\mathcal{P}(n, n-r)$ the set of permutations of n numbers taken $n-r$ at a time. Let $\sigma \in \mathcal{P}(n, n-r)$. If $\sigma(i) > i$, then we say that σ has an excedance at position i , where $1 \leq i \leq n-r$. The *r -restricted Eulerian number*, denoted by $\langle n \rangle_{k,r}$, is defined as the number of permutations in $\mathcal{P}(n, n-r)$ having k excedances (see [12, A144696, A144697, A144698, A144699] for details).

A *Stirling permutation* of order n is a permutation of the multiset $\{1, 1, 2, 2, \dots, n, n\}$ such that for each i , $1 \leq i \leq n$, the elements lying between the two occurrences of i are greater than i . The *second-order Eulerian number* $\langle\langle n \rangle\rangle_k$ is the number of Stirling permutation of order n with k ascents (see [12, A008517]). The combinatorial interpretations for the second-order Eulerian numbers $\langle\langle n \rangle\rangle_k$ have been extensively investigated (see [1, 8, 9]). It is well known that the numbers $\langle\langle n \rangle\rangle_k$ satisfy the recurrence relation

$$\langle\langle n+1 \rangle\rangle_k = (2n-k+1) \langle\langle n \rangle\rangle_{k-1} + (k+1) \langle\langle n \rangle\rangle_k, \quad (2)$$

with initial condition $\langle\langle 1 \rangle\rangle_0 = 1$ and boundary conditions $\langle\langle n \rangle\rangle_k = 0$ for $n \leq k$ or $k < 0$ (see [12, A008517]).

Let B_n denote the set of signed permutations of $\pm[n]$ such that $\pi(-i) = -\pi(i)$ for all i , where $\pm[n] = \{\pm 1, \pm 2, \dots, \pm n\}$. Let

$$B_n(x) = \sum_{k=0}^n B(n, k) x^k = \sum_{\pi \in B_n} x^{\text{des}_B(\pi)},$$

where

$$\text{des}_B = |\{i \in [n] : \pi(i-1) > \pi(i)\}|$$

with $\pi(0) = 0$. The polynomial $B_n(x)$ is called an *Eulerian polynomial of type B*, while $B(n, k)$ is called an *Eulerian number of type B* (see [12, A060187]). The first few of these polynomials are listed below:

$$B_0(x) = 1, B_1(x) = 1 + x, B_2(x) = 1 + 6x + x^2, B_3(x) = 1 + 23x + 23x^2 + x^3.$$

The numbers $B(n, k)$ satisfy the recurrence relation

$$B(n+1, k) = (2n - 2k + 3)B(n, k-1) + (2k+1)B(n, k), \quad (3)$$

with initial condition $B(0, 0) = 1$ and boundary conditions $B(0, k) = 0$ for $k \geq 1$. An explicit formula for $B(n, k)$ is given as follows:

$$B(n, k) = \sum_{i=0}^k (-1)^i \binom{n+1}{i} (2k - 2i + 1)^n$$

for $0 \leq k \leq n$ (see [7] for details).

The *unsigned Stirling number of the first kind* $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$ is the number of permutations in \mathcal{S}_n with exactly k cycles (see [12, A132393]). The *Stirling number of the second kind* $\left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\}$ is the number of ways to partition $[n]$ into k blocks (see [12, A008277]). Let $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$ denote the number of ways to partition $[n]$ into k nonempty linearly ordered subsets. The numbers $\left[\begin{smallmatrix} n \\ k \end{smallmatrix} \right]$ are called *the unsigned Lah numbers* (see [12, A105278]).

We recall some known results on context-free grammars.

Proposition 1 ([2, Eq. 4.8]). *If $G = \{x \rightarrow xy, y \rightarrow y\}$, then*

$$D^n(x) = x \sum_{k=1}^n \left\{ \begin{smallmatrix} n \\ k \end{smallmatrix} \right\} y^k.$$

Proposition 2 ([5, Section 2.1]). *If $G = \{x \rightarrow xy, y \rightarrow xy\}$, then*

$$D^n(x) = x \sum_{k=0}^{n-1} \left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle x^k y^{n-k}.$$

Proposition 3 ([4]). *If $G = \{x \rightarrow x^2y, y \rightarrow x^2y\}$, then*

$$D^n(x) = \sum_{k=0}^{n-1} \left\langle\left\langle \begin{smallmatrix} n \\ k \end{smallmatrix} \right\rangle\right\rangle x^{2n-k} y^{k+1}.$$

Proposition 4 ([11]). *If $G = \{x \rightarrow y^2, y \rightarrow xy\}$, then*

$$D^n(x) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} W_{n,k} x^{n-2k-1} y^{2k+2}, \quad D^n(y) = \sum_{k=0}^{\lfloor n/2 \rfloor} W_{n,k}^l x^{n-2k} y^{2k+1},$$

where $W_{n,k}$ is the number of permutations in \mathcal{S}_n with k interior peaks and $W_{n,k}^l$ is the number of permutations in \mathcal{S}_n with k left peaks.

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2 Results

For $n \geq 0$, we always assume that

$$(xD)^{n+1}(x) = (xD)(xD)^n(x) = xD((xD)^n(x)).$$

The following theorem is in a sense “dual” to Proposition 3.

Theorem 5. *If $G = \{x \rightarrow xy, y \rightarrow xy\}$, then*

$$(xD)^n(x) = \sum_{k=0}^{n-1} \left\langle\!\left\langle n \right\rangle\!\right\rangle_k x^{2n-k} y^{k+1} \quad \text{for } n \geq 1.$$

Proof. For $n \geq 1$, we define

$$(xD)^n(x) = \sum_{k=0}^{n-1} E(n, k) x^{2n-k} y^{k+1}. \quad (4)$$

Note that

$$(xD)(x) = x^2 y, (xD)(x^2 y) = x^4 y + 2x^3 y^2.$$

Then $E(1, 0) = \left\langle\!\left\langle 1 \right\rangle\!\right\rangle_0 = 1$, $E(2, 0) = \left\langle\!\left\langle 2 \right\rangle\!\right\rangle_0 = 1$ and $E(2, 1) = \left\langle\!\left\langle 2 \right\rangle\!\right\rangle_1 = 2$. Using (4), we obtain

$$(xD)(xD)^n(x) = \sum_{k=0}^{n-1} (2n-k) E(n, k) x^{2n-k+1} y^{k+2} + \sum_{k=0}^{n-1} (k+1) E(n, k) x^{2n-k+2} y^{k+1}.$$

Therefore,

$$E(n+1, k) = (2n-k+1) E(n, k-1) + (k+1) E(n, k).$$

Comparing with (2), we see that the coefficients $E(n, k)$ satisfy the same recurrence and initial conditions as $\left\langle\!\left\langle n \right\rangle\!\right\rangle_k$, so they agree. \square

Now we present the main result of this paper.

Theorem 6. *For $n \geq 1$, we have the the following results:*

(c₁) *If $G = \{x \rightarrow xy^2, y \rightarrow x^2 y\}$, then*

$$D^n(xy) = xy \sum_{k=0}^n B(n, k) x^{2n-2k} y^{2k}.$$

(c₂) *If $G = \{x \rightarrow xy^2, y \rightarrow x^2 y\}$, then*

$$D^n(x^2 y^2) = 2^n x^2 y^2 \sum_{k=0}^n \left\langle\!\left\langle n+1 \right\rangle\!\right\rangle_k x^{2n-2k} y^{2k}.$$

(c₃) *If $G = \{x \rightarrow xy^2, y \rightarrow x^2 y\}$, then*

$$D^n(x) = x \sum_{k=1}^n N(n, k) x^{2n-2k} y^{2k},$$

where the number $N(n, k)$ enumerates perfect matchings of $[2n]$ with the restriction that only k matching pairs have odd smaller entries (see [12, A185411]).

(c₄) If $G = \{x \rightarrow xy, y \rightarrow xy\}$, then

$$D^n(xy^r) = x \sum_{k=0}^n \left\langle \begin{matrix} n+r \\ k \end{matrix} \right\rangle_r x^k y^{n+r-k}.$$

(c₅) If $G = \{x \rightarrow xy^2, y \rightarrow xy\}$, then

$$D^n(x) = x \sum_{k=0}^{n-1} 2^k \left\langle \begin{matrix} n \\ k \end{matrix} \right\rangle x^k y^{2n-2k}.$$

(c₆) Consider the numbers $T(n, k)$ with generating function

$$\sqrt{A(2t, z)} = 1 + \sum_{n \geq 1} \sum_{k=1}^n T(n, k) t^k \frac{z^n}{n!},$$

where $A(t, z)$ is given by (1) (see [12, A156920]). If $G = \{x \rightarrow xy^2, y \rightarrow xy\}$, then

$$D^n(y) = \sum_{k=1}^n T(n, k) x^k y^{2n-2k+1}.$$

(c₇) If $G = \{x \rightarrow x^2y, y \rightarrow y\}$, then

$$D^n(x) = x \sum_{k=1}^n k! \left\{ \begin{matrix} n \\ k \end{matrix} \right\} x^k y^k.$$

(c₈) If $G = \{x \rightarrow x^2y, y \rightarrow y^2\}$, then

$$D^n(x) = x \sum_{k=1}^n k! \left[\begin{matrix} n \\ k \end{matrix} \right] x^k y^n.$$

(c₉) If $G = \{x \rightarrow xy^2, y \rightarrow y^2\}$, then

$$D^n(x) = x \sum_{k=1}^n \left[\begin{matrix} n \\ k \end{matrix} \right] y^{n+k}.$$

(c₁₀) If $G = \{x \rightarrow xy^3, y \rightarrow y^3\}$, then

$$D^n(x) = x \sum_{k=1}^n b(n, k) y^{2n+k},$$

where $b(n, k)$ is the number of forests with k rooted ordered trees with n non-root vertices labeled in an organic way (see [12, A035342]).

(c₁₁) For a fixed positive integer $r \geq 4$, if $G = \{x \rightarrow xy^r, y \rightarrow y^r\}$, then

$$D^n(x) = x \sum_{k=1}^n a(n, k; r) y^{(r-1)n+k},$$

where $a(n, k; r)$ enumerates unordered n -vertex k -forests composed of k plane increasing r -ary trees (see [12, A035469, A049029, A049385, A092082]).

(c₁₂) If $G = \{x \rightarrow x^2y, y \rightarrow xy\}$, then

$$D^n(y) = x^n \sum_{k=1}^n d(n, k) y^k,$$

where $d(n, k)$ is the number of increasing mobiles (circular rooted trees) with n nodes and k leaves (see [12, A055356]).

Proof. We only prove (c₁) and the others can be proved in a similar way. Note that $D(x) = xy^2$ and $D(y) = x^2y$. Then

$$D(xy) = xy(x^2 + y^2), D^2(xy) = D(D(xy)) = xy(x^4 + 6x^2y^2 + y^4).$$

For $n \geq 1$, we define

$$D^n(xy) = xy \sum_{k=0}^n G(n, k) x^{2n-2k} y^{2k}.$$

Hence $G(1, 0) = B(1, 0)$ and $G(1, 1) = B(1, 1)$. Since

$$D^{n+1}(xy) = D(D^n(xy)) = \sum_{k=0}^n (2n-2k+1)G(n, k)x^{2n-2k+1}y^{2k+3} + \sum_{k=0}^n (2k+1)G(n, k)x^{2n-2k+3}y^{2k+1},$$

there follows

$$G(n+1, k) = (2n-2k+3)G(n, k-1) + (2k+1)G(n, k).$$

It follows from (3) that $G(n, k)$ satisfies the same recurrence and initial conditions as $B(n, k)$, so they agree. \square

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